Homotopic Maps to S^1 Have Homeomorphic Mapping Swirls, and Consequence for Pseudo-Spines of 4-Manifolds

by

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We call a compact contractible n-manifold a *homotopy n-ball*. A subset X of the interior of a manifold M is called a *(topological) spine* of M if M is homeomorphic to the mapping cylinder of a map from ∂M to X. X is called a *pseudo-spine* of M if M–X is homeomorphic to $\partial M \times [0,\infty)$.

In [1] it is proved that for $n \geq 5$, every homotopy n-ball has a wild arc spine. It is observed, however, that in general homotopy 4-balls don't have arc spines. In fact, a homotopy 4-ball with an arc spine must be either a 4-ball or the cone over a non-trivial homotopy 3-sphere (if one exists). Thus, a homotopy 4-ball with a non-simply connected boundary can't have an arc spine.

The Mazur 4-manifold [6] is a homotopy 4-ball with a non-simply connected boundary. It is a celebrated consequence of [5] and [3] that the Mazur 4-manifold has an arc pseudo-spine.

The naively optimistic conjecture motivating this paper is: every homotopy 4-ball has an arc pseudo-spine. We will reinterpret and generalize the method of [5] through the introduction of the mapping swirl construction. We will prove several theorems about mapping swirls which allow us to produce canonical pseudo-spines for a special class of compact 4-manifolds which includes the Mazur 4-manifold. (This class of compact 4-manifolds consists of all those obtained by attaching finitely many 2-handles to $B^3 \times S^1$.) We will then speculate about the possibility of finding simple pseudo-spines for all compact 4-manifolds and, in particular, for homotopy 4-balls.

1. Motivation: the Psuedo-Spine of the Mazur 4-Manifold

We briefly sketch the proof that the Mazur 4-manifold has an arc pseudo-spine to motivate subsequent developments. The Mazur 4-manifold M^4 is obtained by attaching a 2-handle to $B^3 \times S^1$ along a curve J in $\partial B^3 \times S^1$. Corresponding to this description, one finds that M^4 has a "dunce hat" spine which is the union of the disk D^2 which is the core of the 2-handle and the mapping cylinder $\mathrm{Cyl}(\pi|J)$ of the restriction to J of the natural projection $\pi : B^3 \times S^1 \to \{0\} \times S^1$ of $B^3 \times S^1$ onto its core. (See Figure 1.) Now in the dunce hat spine, replace the mapping cylinder $\mathrm{Cyl}(\pi|J)$ by the "mapping swirl" $\mathrm{Swl}(\pi|J)$ in which the fiber emanating from a point p on J, instead of running straight from p to its image $\pi(p)$ in $\{0\} \times S^1$, spirals infinitely in the S^1-direction in $B^3 \times S^1$ as it approaches $\{0\} \times S^1$. (See Figure 2.) The resulting object $D^2 \cup \mathrm{Swl}(\pi|J)$ is, according to [5], a pseudo-spine of

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M^4 and (remarkably) a topological disk wildly embedded in M^4. (See Figure 2.) Thus, M^4 has a disk pseudo-spine. The result of [3] then allows us to "squeeze" this disk to an arc to conclude that M^4 has an arc pseudo-spine.

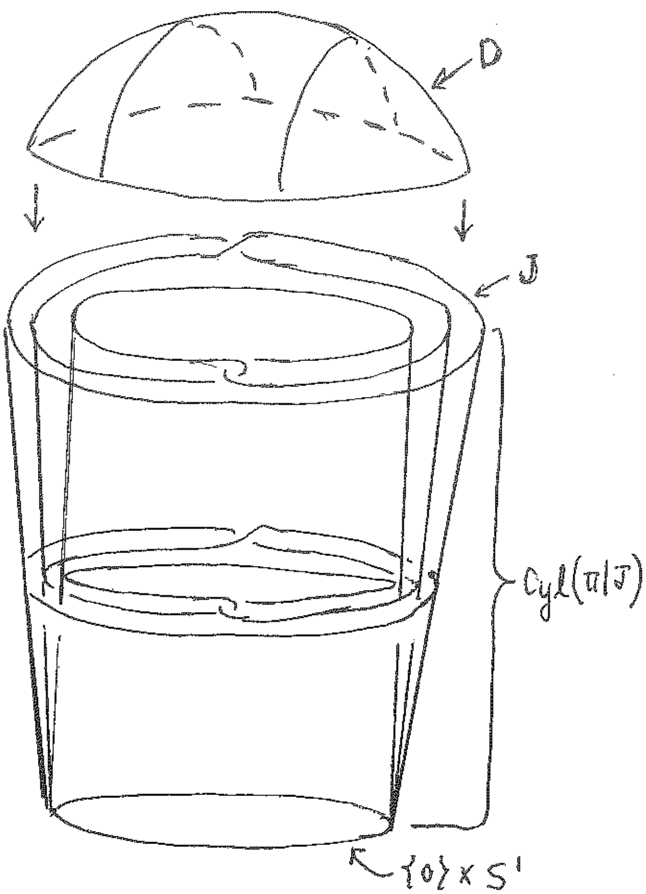

Figure 1

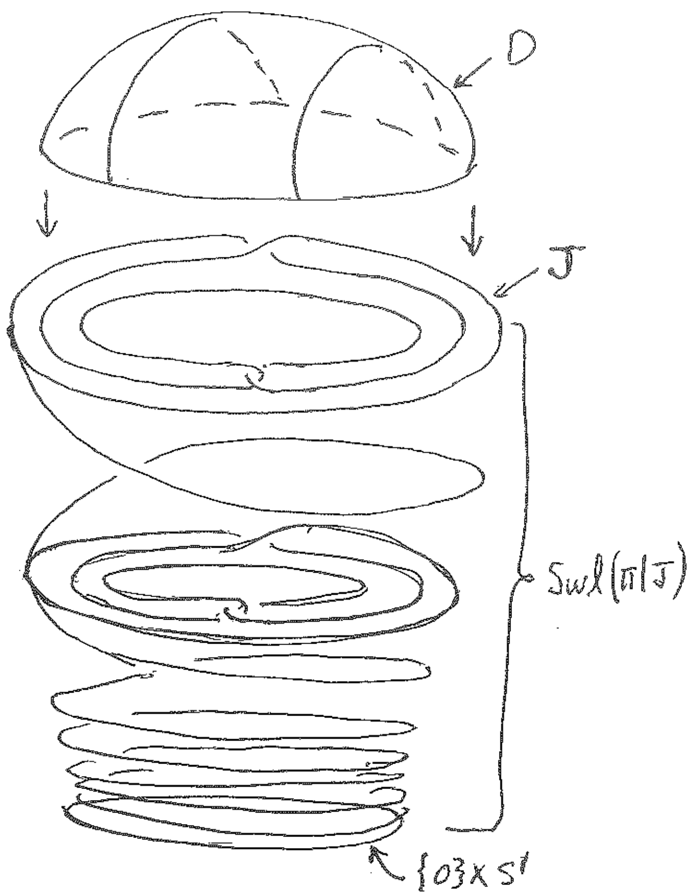

Figure 2

2. The Mapping Swirl Construction: Definitions and Statements of Results

We will now give a formal definition of the object which we called the "mapping swirl" in the preceding section. For a space X, we define the *cone* on X, denoted C(X), to be the quotient space $[0,\infty]\times X/\{\infty\}\times X$. (It is convenient for our purposes to make this slightly non-standard definition of C(X) instead of the more usual $[0,1]\times X/\{1\}\times X$.) For $(t,x) \in [0,\infty]\times X$, we let tx denote the corresponding point of C(X); and we let ∞ denote the point of C(X) corresponding to $\{\infty\}\times X$. We similarly define the *suspension* of X, denoted $\Sigma(X)$, to be the quotient space $[-\infty,\infty]\times X/\{\{-\infty\}\times X,\{\infty\}\times X\}$; and for $(t,x) \in [-\infty,\infty]\times X$, we let tx denote the corresponding point of $\Sigma(X)$; and we let $\pm\infty$ denote the points of $\Sigma(X)$ corresponding to $\{\pm\infty\}\times X$.

Let $f : X \to Y$ be a map. To motivate our upcoming definition of Swl(f), we observe that Cyl(f), the mapping cylinder of f, naturally embeds in $C(X)\times Y$. We regard Cyl(f) as the quotient space $X\times[0,\infty]\cup Y/\sim$ where $\sim$ identifies (x,∞) with $f(x)$ for $x \in X$. To embed Cyl(f) in $C(X)\times Y$, we identify the equivalence class $[(x,t)] \in \mathrm{Cyl}(f)$ of the point $(x,t) \in X\times[0,\infty)$ with the point $(tx,f(x)) \in C(X)\times Y$, and we identify the equivalence class $[y] \in \mathrm{Cyl}(f)$ of the point $y \in Y$ with the point $(\infty,y) \in C(X)\times Y$. In other words, Cyl(f) is identified with the subset

$$\{ (tx,f(x)) \in C(X)\times Y : (t,x) \in [0,\infty)\times X \} \cup (\{\infty\}\times Y)$$

of $C(X)\times Y$. A similar observation reveals that the double mapping cylinder of f, DblCyl(f), naturally embeds in $\Sigma(X)\times Y$ as the subset

$$\{ (tx,f(x)) \in \Sigma(X)\times Y : (t,x) \in (-\infty,\infty)\times X \} \cup (\{-\infty\}\times Y) \cup (\{\infty\}\times Y).$$

Now consider a map $f : X \to S^1$. Define the *mapping swirl* of f, denoted Swl(f) to be the subset

$$\{ (tx,e^{2\pi i t}f(x)) \in C(X)\times S^1 : (t,x) \in [0,\infty)\times X \} \cup (\{\infty\}\times S^1)$$

of $C(X)\times S^1$. Similarly we define the *double mapping swirl* of f, denoted DblSwl(f) to be the subset

$$\{ (tx,e^{2\pi i t}f(x)) \in \Sigma(X)\times S^1 : (t,x) \in (-\infty,\infty)\times X \} \cup (\{-\infty,\infty\}\times S^1).$$

For each integer n, let $\phi_n : S^1 \to S^1$ denote the map $\phi_n(z) = z^n$; and let X(n) denote the adjunction space $B^2\cup_{\phi_n}S^1$. Thus, $X(\pm1)$ is a disk, and $X(\pm2)$ is a projective plane. For integers $n_1, n_2, \cdots, n_k$, let $X(n_1, n_2, \cdots, n_k)$ denote the adjunction space $(B^2\times\{1,2,\cdots,k\})\cup_\Phi S^1$ where $\Phi : S^1\times\{1,2,\cdots,k\} \to S^1$ is the map defined by $\Phi(z,i) = \phi_{n_i}(z)$ for $z \in S^1$ and $1 \le i \le k$. Thus, $X(n_1, n_2, \cdots, n_k)$ is the union of the spaces $X(n_1)$, $X(n_2)$, $\cdots$, $X(n_k)$ with all of their natural S^1 subsets identified.

We now state our main results.

Theorem 1. If X is a compact metric space, and $f, g : X \to S^1$ are homotopic maps, then Swl(f) is homeomorphic to Swl(g).

Theorem 2. If X is a compact metric space, n is a non-zero integer, and $f : X\times S^1 \to S^1$ is the map $f(x,z) = z^n$, then Swl(f) is homeomorphic to Cyl(f).

Corollary 1. If X is a compact metric space, n is a non-zero integer, f, g : $X \times S^1 \to S^1$ are maps such that f is homotopic to g and $g(x,z) = z^n$, then Swl(f) is homeomorphic to Cyl(g).

Corollary 2. If f : $S^1 \to S^1$ is a degree $n \neq 0$ map, then Swl(f) is homeomorphic to Cyl($z \mapsto z^n$). In particular, Swl(f) is an annulus if $n = \pm 1$, and Swl(f) is a Mobius strip if $n = \pm 2$.

Theorem 3. Suppose $C_1, C_2, \cdots, C_k$ are disjoint 1-spheres in $(\partial B^3) \times S^1$, and $M^4 = (B^3 \times S^1) \cup (H_1 \cup H_2 \cup \cdots \cup H_k)$ where H_i is a 2-handle attached to $B^3 \times S^1$ along C_i, for $1 \leq i \leq k$. Let $\pi : \partial B^3 \times S^1 \to S^1$ denote the projection map, and let n_i denote the degree of the map $\pi | C_i : C_i \to S^1$ for $1 \leq i \leq k$. Then M^4 has a pseudo-spine homeomorphic to $X(n_1, n_2, \cdots, n_k)$.

Corollary 3. Suppose C is a 1-sphere in $(\partial B^3) \times S^1$, and $M^4 = (B^3 \times S^1) \cup H$ where H is a 2-handle attached to $B^3 \times S^1$ along C. Let $\pi : B^3 \times S^1 \to S^1$ denote the projection map, and suppose that the map $\pi | C : C \to S^1$ is degree one. Then M^4 has an arc pseudo-spine.

Observe that Corollary 3 includes the fact that the Mazur 4-manifold has an arc pseudo-spine.

3. Sketches of Proofs of Theorems

Proof of Theorem 1. Let f, g : $X \to S^1$ be homotopic maps.

Step 1. DblSwl(f) is homeomorphic to DblSwl(g).

Observe that DblSwl(f) = $(\cup_{x \in X} \mathfrak{F}(x)) \cup (\{-\infty,\infty\} \times S^1)$ where for each $x \in X$, $\mathfrak{F}(x)$ is the "fiber" $\{ (tx, e^{2\pi i t} f(x)) : t \in (-\infty,\infty) \}$ of DblSwl(f) which lies in $((-\infty,\infty)x) \times S^1 \subset (\Sigma(X)) \times S^1$. Similarly, DblSwl(g) = $(\cup_{x \in X} \mathfrak{G}(x)) \cup (\{-\infty,\infty\} \times S^1)$ where for each $x \in X$, $\mathfrak{G}(x)$ is the "fiber" $\{ (tx, e^{2\pi i t} g(x)) : t \in (-\infty,\infty) \}$ of DblSwl(g) which lies in $((-\infty,\infty)x) \times S^1$. For $x \in X$, in $((-\infty,\infty)x) \times S^1 \subset (\Sigma(X)) \times S^1$, we regard $(-\infty,\infty)x$ as the "vertical" direction and S^1 as the "horizontal" direction. We will describe a homeomorphism Φ of $(\Sigma(X)) \times S^1$ which carries DblSwl(f) to DblSwl(g). Φ restricts to the identity on $\{-\infty,\infty\} \times S^1$. For each $x \in X$, Φ carries $((-\infty,\infty)x) \times S^1$ onto itself; and within $((-\infty,\infty)x) \times S^1$, Φ is a "vertical" shift that moves $\mathfrak{F}(x)$ onto $\mathfrak{G}(x)$. (See Figure 3.) Explicitly, there is a vertical shift function $\sigma : X \to (-\infty,\infty)$ such that $\Phi(tx,z) = ((t+\sigma(x))x,z)$ for $(t,x) \in (-\infty,\infty) \times X$ and $z \in S^1$. σ is essentially determined by lifting the homotopy joining f to g in S^1 to a homotopy in $(-\infty,\infty)$.

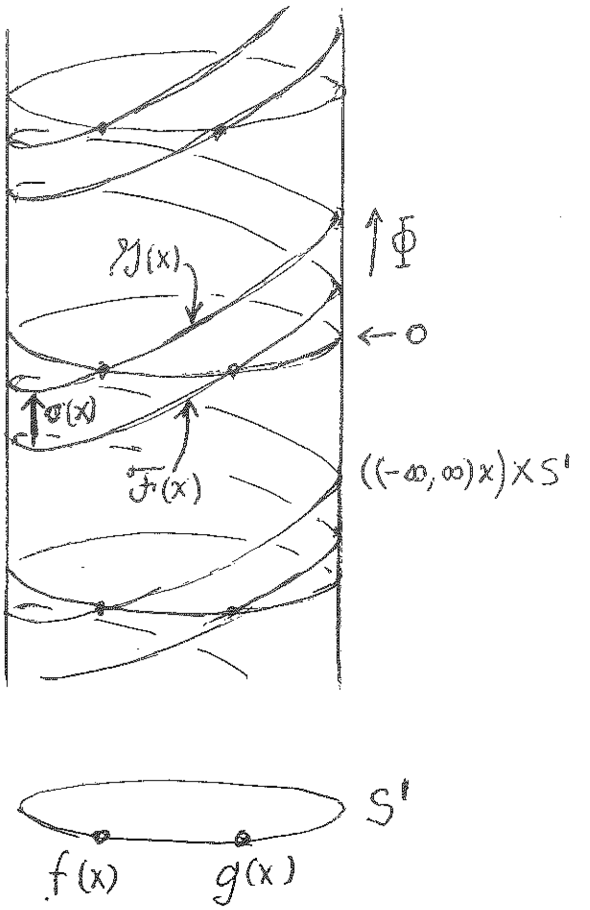

Figure 3

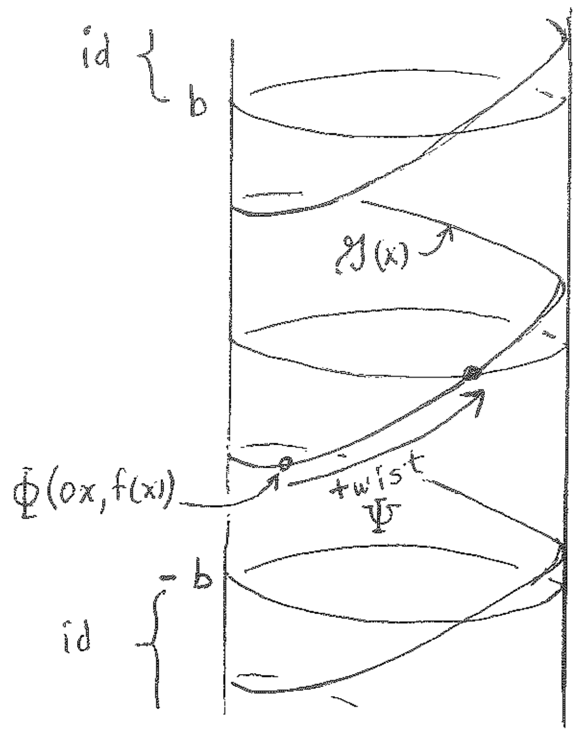

Figure 4

In more detail: suppose $h : X\times[0,1] \to S^1$ is a homotopy such that $h(x,0) = g(x)$ and $h(x,1) = f(x)$. We exploit the fact that S^1 is a group under complex multiplication to define the map $k : X\times[0,1] \to S^1$ by $k(x,t) = h(x,t)/h(x,0)$. Thus, $k(x,0) = 1$ and $k(x,1)g(x) = f(x)$ for $x \in X$. Let $e : (-\infty,\infty) \to S^1$ denote the exponential covering map $e(t) = e^{2\pi it}$. Let $\tilde{k} : X\times[0,1] \to (-\infty,\infty)$ be the lift of k (i.e., $e\circ\tilde{k} = k$) such that $\tilde{k}(x,0) = 0$ for all $x \in X$. Define $\sigma : X \to (-\infty,\infty)$ by $\sigma(x) = \tilde{k}(x,1)$. Observe that for each $x \in X$, $f(x)/e^{2\pi i\sigma(x)} = f(x)/e(\tilde{k}(x,1)) = f(x)/k(x,1) = g(x)$. It is now straightforward to verify that a homeomorphism Φ of $(\Sigma(X))\times S^1$ is defined by setting $\Phi(tx,z) = ((t+\sigma(x))x,z)$ for $(t,x) \in (-\infty,\infty)\times X$ and $z \in S^1$, and by requiring that $\Phi|\{-\infty,\infty\}\times S^1 = id$. To verify that Φ(DblSwl(f)) = DblSwl(g), one shows that $\Phi(\mathfrak{F}(x)) = \mathcal{G}(x)$ for each $x \in X$. To this end, consider a typical point $(tx,e^{2\pi it}f(x))$ lying in the fiber $\mathfrak{F}(x)$ of DblSwl(f). Φ moves this point to the point $((t + \sigma(x))x,e^{2\pi it}f(x)) = ((t + \sigma(x))x,e^{2\pi i(t + \sigma(x))}f(x)/e^{2\pi i\sigma(x)}) = ((t + \sigma(x))x,e^{2\pi i(t + \sigma(x))}g(x))$ which is a point of the fiber $\mathcal{G}(x)$ of DblSwl(g).

Step 2. Swl(f) is homeomorphic to Swl(g).

Observe that $Swl(f) = (\cup_{x\in X}\mathfrak{F}^+(x)) \cup (\{\infty\}\times S^1)$ where for each $x \in X$, $\mathfrak{F}^+(x)$ is the "fiber" $\{ (tx,e^{2\pi it}f(x)) : t \in [0,\infty) \}$ of Swl(f) which lies in $([0,\infty)x)\times S^1 \subset (C(X))\times S^1$. Similarly, $Swl(g) = (\cup_{x\in X}\mathfrak{G}^+(x)) \cup (\{\infty\}\times S^1)$ where for each $x \in X$, $\mathfrak{G}^+(x)$ is the "fiber" $\{ (tx,e^{2\pi it}g(x)) : t \in [0,\infty) \}$ of Swl(g) which lies in $([0,\infty)x)\times S^1$. For $x \in X$, since $\mathfrak{F}^+(x) \subset \mathfrak{F}(x)$, then $\Phi(\mathfrak{F}^+(x)) \subset \Phi(\mathfrak{F}(x)) \subset \mathfrak{G}(x)$; also $\mathfrak{G}^+(x) \subset \mathfrak{G}(x)$. We will describe a homeomorphism Ψ of $(\Sigma(X))\times S^1$ which carries $\Phi(Swl(f))$ onto Swl(g). Ψ combines a vertical shift with a horizontal twist in a corkscrew motion which, for each $x \in X$, carries $\mathfrak{G}(x)$ onto itself and moves $\Phi(\mathfrak{F}^+(x))$ onto $\mathfrak{G}^+(x)$. Also Ψ restricts to the identity on a neighborhood of $\{-\infty,\infty\}\times S^1$. (See Figure 4.)

Since X is compact, there is a $b \in (0,\infty)$ such that $\sigma(X) \subset (-b,b)$. It is easy to give a formula for a map $\tau : [-\infty,\infty]\times X \to [-\infty,\infty]$ such that for each $x \in X$, $t \mapsto \tau(t,x) : [-\infty,\infty] \to [-\infty,\infty]$ is an order preserving homeomorphism which restricts to the identity on $[-\infty,-b]\cup[b,\infty]$ and which carries $\sigma(x)$ to 0. (Thus, $\tau(\sigma(x),x) = 0$, and $\tau(t,x) = t$ if $|t| \geq b$.) Now define a homeomorphism Ψ of $(\Sigma(X))\times S^1$ by $\Psi(tx,z) = (\tau(t,x)x, e^{2\pi i(\tau(t,x) - t)}z)$ for $(tx,z) \in (\Sigma(X))\times S^1$. Clearly, $\Psi(tx,z) = (tx,z)$ for $|t| \geq b$; so Ψ restricts to the identity on $\{-\infty,\infty\}\times S^1$. For $x \in X$, one easily computes that Ψ moves the typical point $(tx,e^{2\pi it}g(x))$ of $\mathfrak{G}(x)$ to the point $(\tau(t,x)x,e^{2\pi i\tau(t,x)}g(x))$ which also belongs to $\mathfrak{G}(x)$; so $\Psi(\mathfrak{G}(x)) = \mathfrak{G}(x)$. Finally, for $x \in X$, $\Psi(\Phi(0x,f(x))) = \Psi(\sigma(x)x,f(x)) = (0x,e^{2\pi i(0 - \sigma(x))}f(x)) = (0x,f(x)/e^{2\pi i\sigma(x)}) = (0x,g(x))$. Thus, $\Psi\circ\Phi$ is an order preserving homeomorphism from $\mathfrak{F}(x)$ onto $\mathfrak{G}(x)$ which carries the boundary point of $\mathfrak{F}^+(x)$ to the boundary point of $\mathfrak{G}^+(x)$. Consequently, $\Psi\circ\Phi(\mathfrak{F}^+(x)) = \mathfrak{G}^+(x)$. It follows that $\Psi\circ\Phi(Swl(f)) = Swl(g)$. □

Proof of Theorem 2. $f : X\times S^1 \to S^1$ satisfies $f(x,z) = z^n$ where $n \neq 0$. Both Cyl(f) and Swl(f) can be regarded as subsets of $C(X\times S^1)\times S^1$. We will describe a homeomorphism Φ of $C(X\times S^1)\times S^1$ which carries Cyl(f) onto Swl(f). Φ achieves this result by twisting in the S^1-direction in the $C(X\times S^1)$ factor of $C(X\times S^1)\times S^1$. For $t(x,z) \in C(X\times S^1)$ and $w \in S^1$, set $\Phi(t(x,z),w) = (t(x,e^{-2\pi it/n}z),w)$ and $\Phi(\infty,w) = (\infty,w)$. This clearly defines a homeomorphism of $C(X\times S^1)\times S^1$. Let $(x,z) \in X\times S^1$ and consider a typical point $(t(x,z),f(x,z)) = (t(x,z),z^n)$ of the fiber emanating from (x,z) in Cyl(f). Set $z' = e^{-2\pi it/n}z$. Then $\Phi(t(x,z),f(x,z)) = (t(x,e^{-2\pi it/n}z),z^n) = (t(x,e^{-2\pi it/n}z),e^{2\pi it}(e^{-2\pi it/n}z)^n) = (t(x,e^{-2\pi it/n}z),e^{2\pi it}f(x,e^{-2\pi it/n}z)) = (t(x,z'),e^{2\pi it}f(x,z'))$ which is a typical point of the fiber $\mathfrak{F}^+(x,z')$ in Swl(f). It follows that $\Phi(Cyl(f)) = Swl(f)$. □

Corollaries 1 and 2 are obvious consequences of Theorems 1 and 2.

Proof of Theorem 3. Recall that $\pi : \partial B^3 \times S^1 \to S^1$ denotes the projection map. Clearly $Cyl(\pi)$ is homeomorphic to $B^3 \times S^1$. On the other hand, since $\pi(x,z) = z$, then by Theorem 2, $Cyl(\pi)$ is homeomorphic to $Swl(\pi)$. Hence, we identify $B^3 \times S^1$ with $Swl(\pi)$.

Recall that $C_1, C_2, \cdots, C_k$ are disjoint 1-spheres in $(\partial B^3) \times S^1$, and $M^4 = (B^3 \times S^1) \cup (H_1 \cup H_2 \cup \cdots \cup H_k)$ where H_i is a 2-handle attached to $B^3 \times S^1$ along C_i, for $1 \leq i \leq k$. Think of the 2-handle H_i as the product of two 2-dimensional disks D_i and E_i with $d_i \in int(D_i)$ and $e_i \in int(E_i)$ so that $H_i \cap (B^3 \times S^1) = (\partial D_i) \times E_i$ and $C_i = (\partial D_i) \times \{e_i\}$. (Thus, $D_i \times \{e_i\}$ is the "core" and $\{d_i\} \times E_i$ is the "cocore" of H_i.) Recall that $n_i = deg(\pi|C_i)$. Set $X = \bigcup_{i=1}^{k} Swl(\pi| C_i) \cup (D_i \times \{e_i\})$. Then clearly X is homeomorphic to $X(n_1, n_2, \cdots, n_k)$. (M^4 and X are represented schematically in Figure 5.)

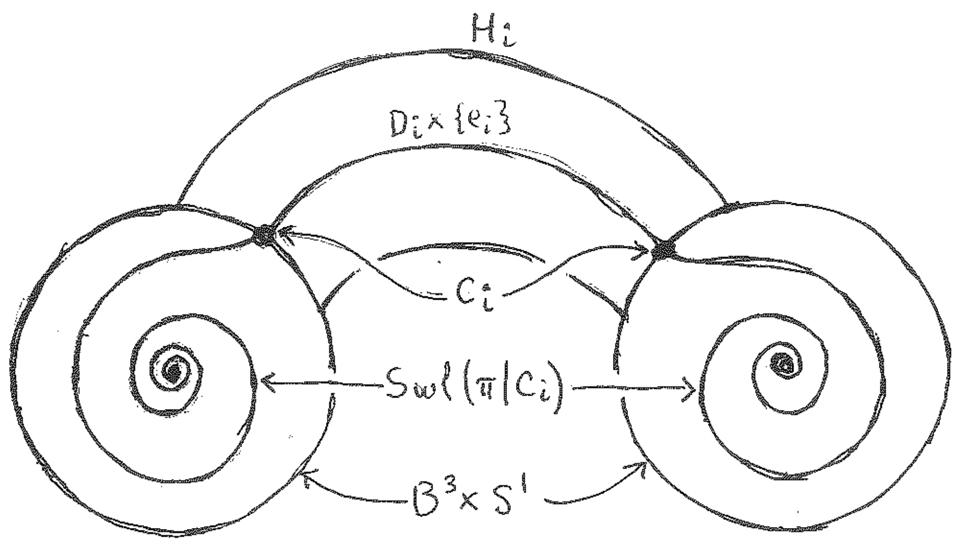

Figure 5

It remains to describe a homeomorphism $h : (\partial M^4) \times [0,\infty) \to M^4 - X$. Note that $(B^3 \times S^1) - X = Swl(\pi) - X$ is the union of the fibers of $Swl(\pi)$ that emanate from the points of $(\partial B^3 \times S^1) - \bigcup_{i=1}^{k} C_i$. Each of these fibers is homeomorphic to $[0,\infty)$. We will "enlarge" this fibering to a fibering of all of $M^4 - X$ by copies of $[0,\infty)$. Let $x \in \partial M^4$. If $x \in \partial B^3 \times S^1$, then $h(\{x\} \times [0,\infty))$ is the fiber of $Swl(\pi)$ emanating from x. If $x \notin \partial B^3 \times S^1$, then $x \in (int(D_i)) \times (\partial E_i)$ for some i, $1 \leq i \leq k$. If $x \in \{d_i\} \times (\partial E_i)$, then $h(\{x\} \times [0,\infty))$ is the radius of the disk $\{d_i\} \times E_i$ joining the center point (d_i,e_i) to x, minus the center point (d_i,e_i). If $x \in$

$(\text{int}(D_i) - \{d_i\}) \times (\partial E_i)$, then $h(\{x\} \times [0,\infty))$ is the union of a curved arc joining x to a point $y \in (\partial D_i) \times (\text{int}(E_i) - \{e_i\})$ together with the fiber of Swl(π) emanating from y. (See Figure 6.)

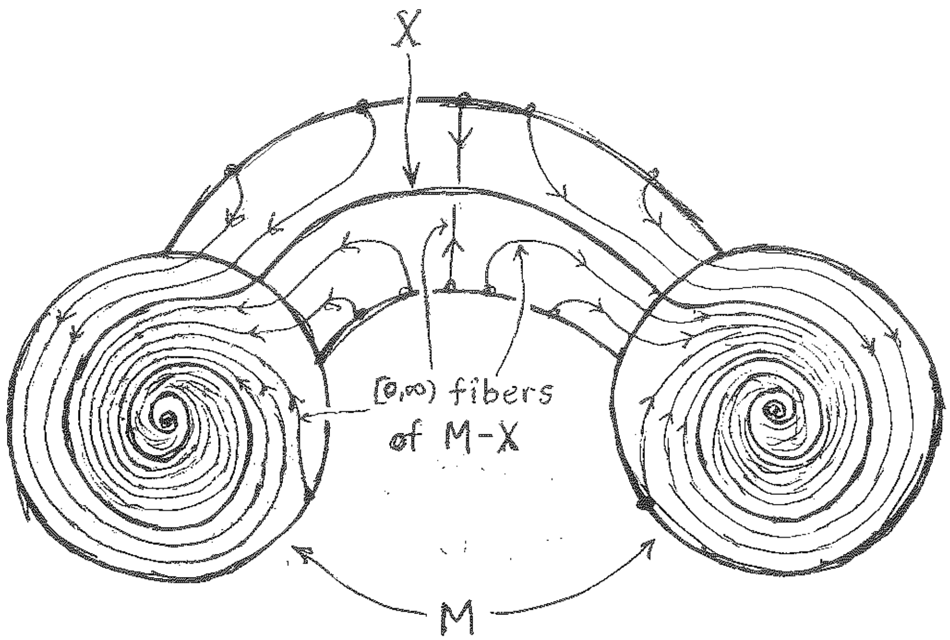

Figure 6

Proof of Corollary 3. Theorem 3 implies that M^4 has a pseudo-spine homeomorphic to X(1). Clearly X(1) is a disk. According to [3], X(1) can be "squeezed" to an arc in int(M^4). (Interpreted literally, [3] applies only in manifolds of dimension 3. However, the methods of [3] work in manifolds of all dimensions ≥ 3. This is fully explained on page 95 of [2].) Thus, M^4 has an arc pseudo-spine. □

4. Conjectures

The techniques and results about pseudo-spines of 4-manifolds presented here are rather modest and restricted. Although at present we have no idea how to enlarge the scope of these results, we are undaunted in formulating conjectures of much greater breadth and boldness.

Conjecture 1. If a compact 4-manifold with is homotopy equivalent to $X(n_1, n_2, \cdots, n_k)$, then it has a pseudo-spine homeomorphic to $X(n_1, n_2, \cdots, n_k)$.

Conjecture 2. Every homotopy 4-ball has an arc pseudo-spine.

We break Conjecture 2 into two conjectures.

Conjecture 2A. Every PL homotopy 4-ball has a handlebody decomposition with no 3- or 4-handles.

Conjecture 2B. If a homotopy 4-ball has a handlebody decomposition with no 3- or 4-handles, then it has an arc pseudo-spine.

Conjecture 3. If two compact 4-manifolds have arc pseudo-spines, then so does their boundary-connected sum.

Conjecture 4. If a compact 4-manifold has a tree pseudo-spine, then it has an arc pseudo-spine.

Conjecture 5. If a compact 4-manifold has a pseudo-spine which is homeomorphic to a 1-dimensional polyhedron, then is has a pseudo-spine which is homeomorphic to a wedge of circles.

Conjecture 6. The 4-ball is the only homotopy 4-ball that has disjoint pseudo-spines.

In connection with Conjecture 6, we note that in [4] it is shown that for $n \geq 9$, there are homotopy n-balls distinct from the n-ball that have disjoint spines. Conjecture 6 asserts that the situation is different in dimension 4.

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